Functional Programming Languages

Thomas Sewell
UNSW

Term 3 2024

Functional Programming

Many languages have been called functional over the years:

Lisp

```
(define (max-of lst)
  (cond
    [(= (length lst) 1) (first lst)]
    [else (max (first lst) (max-of (rest lst)))]))
```

Functional Programming

Many languages have been called functional over the years:

Haskell

$$maxOf :: [\mathrm{Int}] \to \mathrm{Int}$$
$$maxOf = foldr1\ max$$

Lisp

```
(define (max-of lst)
  (cond
    [(= (length lst) 1) (first lst)]
    [else (max (first lst) (max-of (rest lst)))]))
```

Functional Programming

Many languages have been called functional over the years:

Haskell

$$maxOf :: [\mathrm{Int}] \rightarrow \mathrm{Int}$$
$$maxOf = foldr1\ max$$

Lisp

```
(define (max-of lst)
  (cond
    [(= (length lst) 1) (first lst)]
    [else (max (first lst) (max-of (rest lst)))]))
```

JavaScript?

```
function maxOf(arr) {
  var max = arr.reduce(function(a, b) {
    return Math.max(a, b);
  }); }
```

Functional Programming

Many languages have been called functional over the years:

Haskell

$$maxOf :: [\mathrm{Int}] \rightarrow \mathrm{Int}$$
$$maxOf = foldr1\ max$$

Lisp

```
(define (max-of lst)
  (cond
    [(= (length lst) 1) (first lst)]
    [else (max (first lst) (max-of (rest lst)))]))
```

JavaScript?

```
function maxOf(arr) {
  var max = arr.reduce(function(a, b) {
    return Math.max(a, b);
  }); }
```

What do they have in common?

Definitions

Unlike imperative languages, functional programming languages are not very crisply defined.

Attempt at a Definition

A *functional programming language* is a programming language derived from or inspired by the λ-calculus, or derived from or inspired by another functional programming language.

The result? If it has λ in it, you can call it functional.

Definitions

Unlike imperative languages, functional programming languages are not very crisply defined.

Attempt at a Definition

A *functional programming language* is a programming language derived from or inspired by the λ-calculus, or derived from or inspired by another functional programming language.

The result? If it has λ in it, you can call it functional.

In this course, we'll consider *purely functional* languages, which have a much better definition.

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Garbage Collection?

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Garbage Collection?

Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Garbage Collection?
Lisp, 1958

Functions as Values?

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Garbage Collection?
Lisp, 1958

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Garbage Collection?
Lisp, 1958

Polymorphism?

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Garbage Collection?
Lisp, 1958

Polymorphism?
ML, 1973

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Type Inference?

Garbage Collection?
Lisp, 1958

Polymorphism?
ML, 1973

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Type Inference?

ML, 1973

Garbage Collection?

Lisp, 1958

Polymorphism?

ML, 1973

Functions as Values?

Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Type Inference?

ML, 1973

Garbage Collection?

Lisp, 1958

Metaprogramming?

Polymorphism?

ML, 1973

Functions as Values?

Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Type Inference?

ML, 1973

Garbage Collection?

Lisp, 1958

Metaprogramming?

Lisp, 1958

Polymorphism?

ML, 1973

Functions as Values?

Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Type Inference?
ML, 1973

Garbage Collection?
Lisp, 1958

Metaprogramming?
Lisp, 1958

Polymorphism?
ML, 1973

Lazy Evaluation?

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Type Inference?
ML, 1973

Garbage Collection?
Lisp, 1958

Metaprogramming?
Lisp, 1958

Polymorphism?
ML, 1973

Lazy Evaluation?
Miranda, 1985

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Monads?

Type Inference?
ML, 1973

Garbage Collection?
Lisp, 1958

Metaprogramming?
Lisp, 1958

Polymorphism?
ML, 1973

Lazy Evaluation?
Miranda, 1985

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Type Inference?
ML, 1973

Monads?
Haskell, 1991

Garbage Collection?
Lisp, 1958

Metaprogramming?
Lisp, 1958

Polymorphism?
ML, 1973

Lazy Evaluation?
Miranda, 1985

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Monads?
Haskell, 1991

Type Inference?
ML, 1973

Garbage Collection?
Lisp, 1958

Software Transactional Memory?

Metaprogramming?
Lisp, 1958

Polymorphism?
ML, 1973

Lazy Evaluation?
Miranda, 1985

Functions as Values?
Lisp, 1958

Why Study FP Languages?

Think of a major innovation in the area of programming languages.

Monads?
Haskell, 1991

Type Inference?
ML, 1973

Garbage Collection?
Lisp, 1958

Software Transactional Memory?
GHC Haskell, 2005

Metaprogramming?
Lisp, 1958

Polymorphism?
ML, 1973

Lazy Evaluation?
Miranda, 1985

Functions as Values?
Lisp, 1958

Purely Functional Programming Languages

The term *purely functional* has a very crisp definition.

Definition

A programming language is *purely functional* if β-reduction (or evaluation in general) is actually a confluence.

In other words, functions have to be mathematical functions, and free of *side effects*.

Purely Functional Programming Languages

The term *purely functional* has a very crisp definition.

Definition

A programming language is *purely functional* if β-reduction (or evaluation in general) is actually a confluence.
In other words, functions have to be mathematical functions, and free of *side effects*.

Consider what would happen if we allowed effects in a functional language:

$$\begin{aligned} &\mathit{count} = 0; \\ &f\ x\ = \{\mathit{count} := \mathit{count} + x; \mathbf{return}\ \mathit{count}\}; \\ &m = (\lambda y.\ y + y)\ (f\ 3) \end{aligned}$$

If we evaluate $f\ 3$ first, we will get $m = 6$, but if we β-reduce m first, we will get $m = 9$. $\Rightarrow$ not confluent.

Making a Functional Language

We're going to make a language called **MinHS**.

1. Three types of values: integers, booleans, and functions.

Making a Functional Language

We're going to make a language called **MinHS**.

1. Three types of values: integers, booleans, and functions.
2. Static type checking (not inference)

Making a Functional Language

We're going to make a language called **MinHS**.

1. Three types of values: integers, booleans, and functions.
2. Static type checking (not inference)
3. Purely functional (no effects)

Making a Functional Language

We're going to make a language called **MinHS**.

1. Three types of values: integers, booleans, and functions.
2. Static type checking (not inference)
3. Purely functional (no effects)
4. Call-by-value (strict evaluation)

Making a Functional Language

We're going to make a language called **MinHS**.

1. Three types of values: integers, booleans, and functions.
2. Static type checking (not inference)
3. Purely functional (no effects)
4. Call-by-value (strict evaluation)

Something not unlike this will appear in your Assignment 1.

Syntax

$$
\begin{array}{llcl}
\textit{Integers} & n & ::= & \cdots \\
\textit{Identifiers} & f, x & ::= & \cdots \\
\textit{Literals} & b & ::= & \texttt{True} \mid \texttt{False} \\
\textit{Types} & \tau & ::= & \mathrm{Bool} \mid \mathrm{Int} \mid \tau_1 \to \tau_2 \\
\textit{Infix Operators} & \circledast & ::= & \texttt{*} \mid \texttt{+} \mid \texttt{==} \mid \cdots \\
\textit{Expressions} & e & ::= & x \mid n \mid b \mid (e) \mid e_1 \circledast e_2 \\
 & & \mid & \texttt{if } e_1 \texttt{ then } e_2 \texttt{ else } e_3
\end{array}
$$

Syntax

Integers	n	$::=$	$\cdots$
Identifiers	f, x	$::=$	$\cdots$
Literals	b	$::=$	$\texttt{True} \mid \texttt{False}$
Types	τ	$::=$	$\mathrm{Bool} \mid \mathrm{Int} \mid \tau_1 \rightarrow \tau_2$
Infix Operators	$\circledast$	$::=$	$\texttt{*} \mid \texttt{+} \mid \texttt{==} \mid \cdots$
Expressions	e	$::=$	$x \mid n \mid b \mid (e) \mid e_1 \circledast e_2$
		$\mid$	$\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3$
		$\mid$	$e_1\ e_2$

Syntax

Integers	n	$::=$	$\cdots$
Identifiers	f, x	$::=$	$\cdots$
Literals	b	$::=$	$\texttt{True} \mid \texttt{False}$
Types	τ	$::=$	$\mathrm{Bool} \mid \mathrm{Int} \mid \tau_1 \to \tau_2$
Infix Operators	$\circledast$	$::=$	$\texttt{*} \mid \texttt{+} \mid \texttt{==} \mid \cdots$
Expressions	e	$::=$	$x \mid n \mid b \mid (e) \mid e_1 \circledast e_2$
		$\mid$	$\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3$
		$\mid$	$e_1\ e_2$
		$\mid$	$\texttt{recfun}\ f :: (\tau_1 \to \tau_2)\ x = e$
			$\uparrow$ Like λ, but with recursion.

As usual, this is ambiguous concrete syntax. But all the precedence and associativity rules apply as in Haskell. We assume a suitable parser.

Examples

Example (Stupid division by 5)

$$\begin{array}{l}\textbf{recfun}\ \mathit{divBy5} :: (\mathrm{Int} \to \mathrm{Int})\ x = \\ \quad \textbf{if}\ x < 5 \\ \qquad \textbf{then}\ 0 \\ \qquad \textbf{else}\ 1 + \mathit{divBy5}\ (x - 5)\end{array}$$

Example (Average Function)

$$\begin{array}{l}\textbf{recfun}\ \mathit{average} :: (\mathrm{Int} \to (\mathrm{Int} \to \mathrm{Int}))\ x = \\ \quad \textbf{recfun}\ \mathit{avX} :: (\mathrm{Int} \to \mathrm{Int})\ y = \\ \qquad (x + y)\ /\ 2\end{array}$$

As in Haskell, $(\mathit{average}\ 15\ 5) = ((\mathit{average}\ 15)\ 5)$.

We don't need no let

This language is so minimal, it doesn't even need **let** expressions.
How can we do without them?

We don't need no let

This language is so minimal, it doesn't even need **let** expressions. How can we do without them?

$$\textbf{let}\ x :: \tau_1 = e_1\ \textbf{in}\ e_2 :: \tau_2 \quad \equiv \quad (\textbf{recfun}\ f :: (\tau_1 \rightarrow \tau_2)\ x = e_2)\ e_1$$

Abstract Syntax

Moving to first order abstract syntax, we get:

Concrete Syntax	Abstract Syntax
n	(`Num` n)
b	(`Lit` n)

Abstract Syntax

Moving to first order abstract syntax, we get:

Concrete Syntax	Abstract Syntax
n	(`Num` n)
b	(`Lit` n)
if c **then** t **else** e	(`If` c t e)

Abstract Syntax

Moving to first order abstract syntax, we get:

Concrete Syntax	Abstract Syntax
n	(`Num` n)
b	(`Lit` n)
if c **then** t **else** e	(`If` c t e)
e_1 e_2	(`Apply` e_1 e_2)

Abstract Syntax

Moving to first order abstract syntax, we get:

Concrete Syntax	Abstract Syntax
n	$(\texttt{Num}\ n)$
b	$(\texttt{Lit}\ n)$
if c **then** t **else** e	$(\texttt{If}\ c\ t\ e)$
$e_1\ e_2$	$(\texttt{Apply}\ e_1\ e_2)$
recfun $f :: (\tau_1 \to \tau_2)\ x = e$	$(\texttt{Recfun}\ \tau_1\ \tau_2\ f\ x\ e)$

Abstract Syntax

Moving to first order abstract syntax, we get:

Concrete Syntax	Abstract Syntax
n	(Num n)
b	(Lit n)
if c **then** t **else** e	(If c t e)
e_1 e_2	(Apply e_1 e_2)
recfun $f :: (\tau_1 \to \tau_2)\ x = e$	(Recfun τ_1 τ_2 f x e)
x	(Var x)

Abstract Syntax

Moving to first order abstract syntax, we get:

Concrete Syntax	Abstract Syntax
n	(Num n)
b	(Lit n)
if c **then** t **else** e	(If c t e)
e_1 e_2	(Apply e_1 e_2)
recfun $f :: (\tau_1 \rightarrow \tau_2)\ x = e$	(Recfun τ_1 τ_2 f x e)
x	(Var x)

What changes when we move to higher order abstract syntax?

Abstract Syntax

Moving to first order abstract syntax, we get:

Concrete Syntax	Abstract Syntax
n	$(\texttt{Num}\ n)$
b	$(\texttt{Lit}\ n)$
if c **then** t **else** e	$(\texttt{If}\ c\ t\ e)$
$e_1\ e_2$	$(\texttt{Apply}\ e_1\ e_2)$
recfun $f :: (\tau_1 \rightarrow \tau_2)\ x = e$	$(\texttt{Recfun}\ \tau_1\ \tau_2\ f\ x\ e)$
x	$(\texttt{Var}\ x)$

What changes when we move to higher order abstract syntax?

1. `Var` terms go away – we use the meta-language's variables.
2. $(\texttt{Recfun}\ \tau_1\ \tau_2\ f\ x\ e)$ now uses meta-language abstraction: $(\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e))$.

Working Statically with HOAS

To Code

We're going to write code for an AST and pretty-printer for MinHS with HOAS.
Seeing as this requires us to look under abstractions without evaluating the term, we have to extend the AST with special "tag" values.

Static Semantics

To check if a MinHS program is well-formed, we need to check:

1. **Scoping** – all variables used must be well defined
2. **Typing** – all operations must be used on compatible types.

Static Semantics

To check if a MinHS program is well-formed, we need to check:

1. **Scoping** – all variables used must be well defined
2. **Typing** – all operations must be used on compatible types.

Our judgement is an extension of the scoping rules to include types:

Under this context of assumptions

$$\Gamma \vdash e : \tau$$

The expression is assigned this type

The context Γ includes typing assumptions for the variables:

$$x : \mathrm{Int}, y : \mathrm{Int} \vdash (\texttt{Plus}\ x\ y) : \mathrm{Int}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool}}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, \qquad\qquad\qquad\qquad \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, x : \tau_1, \qquad\qquad \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, x : \tau_1, f : (\tau_1 \to \tau_2) \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, x : \tau_1, f : (\tau_1 \to \tau_2) \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) : \tau_1 \to \tau_2}$$

$$\frac{}{\Gamma \vdash (\texttt{Apply}\ e_1\ e_2) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, x : \tau_1, f : (\tau_1 \to \tau_2) \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2}{\Gamma \vdash (\texttt{Apply}\ e_1\ e_2) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, x : \tau_1, f : (\tau_1 \to \tau_2) \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash (\texttt{Apply}\ e_1\ e_2) :}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, x : \tau_1, f : (\tau_1 \to \tau_2) \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash (\texttt{Apply}\ e_1\ e_2) : \tau_2}$$

Static Semantics

$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \mathrm{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash (\texttt{Plus}\ e_1\ e_2) : \mathrm{Int}}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\texttt{If}\ e_1\ e_2\ e_3) : \tau}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash (\texttt{Var}\ x) : \tau} \qquad \frac{\Gamma, x : \tau_1, f : (\tau_1 \to \tau_2) \vdash e : \tau_2}{\Gamma \vdash (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.\ x.\ e)) : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash (\texttt{Apply}\ e_1\ e_2) : \tau_2}$$

Let's implement a *type checker*.

Dynamic Semantics

Structural Operational Semantics (Small-Step)

Initial states:

Dynamic Semantics

Structural Operational Semantics (Small-Step)

Initial states: All well typed expressions.
Final states:

Dynamic Semantics

Structural Operational Semantics (Small-Step)

Initial states: All well typed expressions.

Final states: $(\texttt{Num}\ n)$, $(\texttt{Lit}\ b)$,

Dynamic Semantics

Structural Operational Semantics (Small-Step)

Initial states: All well typed expressions.
Final states: (`Num` n), (`Lit` b), `Recfun` too!

Evaluation of built-in operations:

$$\frac{e_1 \mapsto e_1'}{(\texttt{Plus}\ e_1\ e_2) \mapsto (\texttt{Plus}\ e_1'\ e_2)}$$

(and so on as per arithmetic expressions)

Specifying If

$$\frac{e_1 \mapsto e_1'}{(\texttt{If}\ e_1\ e_2\ e_3) \mapsto (\texttt{If}\ e_1'\ e_2\ e_3)}$$

$$\frac{}{(\texttt{If}\ (\texttt{Lit True})\ e_2\ e_3) \mapsto e_2}$$

$$\frac{}{(\texttt{If}\ (\texttt{Lit False})\ e_2\ e_3) \mapsto e_3}$$

How about Functions?

Recall that `Recfun` is a final state – we don't need to evaluate it unless it's applied to an argument.

Evaluating function application requires us to:

1. Evaluate the left expression to get a `Recfun`;
2. evaluate the right expression to get an argument value; and
3. evaluate the function's body, after supplying substitutions for the abstracted variables.

$$\frac{e_1 \mapsto e_1'}{(\texttt{Apply}\ e_1\ e_2) \mapsto (\texttt{Apply}\ e_1'\ e_2)}$$

$$\frac{e_2 \mapsto e_2'}{(\texttt{Apply}\ (\texttt{Recfun} \ldots)\ e_2) \mapsto (\texttt{Apply}\ (\texttt{Recfun} \ldots)\ e_2')}$$

How about Functions?

Recall that `Recfun` is a final state – we don't need to evaluate it unless it's applied to an argument.

Evaluating function application requires us to:

1. Evaluate the left expression to get a `Recfun`;
2. evaluate the right expression to get an argument value; and
3. evaluate the function's body, after supplying substitutions for the abstracted variables.

$$\frac{e_1 \mapsto e_1'}{(\texttt{Apply}\ e_1\ e_2) \mapsto (\texttt{Apply}\ e_1'\ e_2)}$$

$$\frac{e_2 \mapsto e_2'}{(\texttt{Apply}\ (\texttt{Recfun} \ldots)\ e_2) \mapsto (\texttt{Apply}\ (\texttt{Recfun} \ldots)\ e_2')}$$

$$\frac{v \in F}{(\texttt{Apply}\ (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.x.\ e))\ v) \mapsto e[x := v, f := (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.x.\ e))]}$$